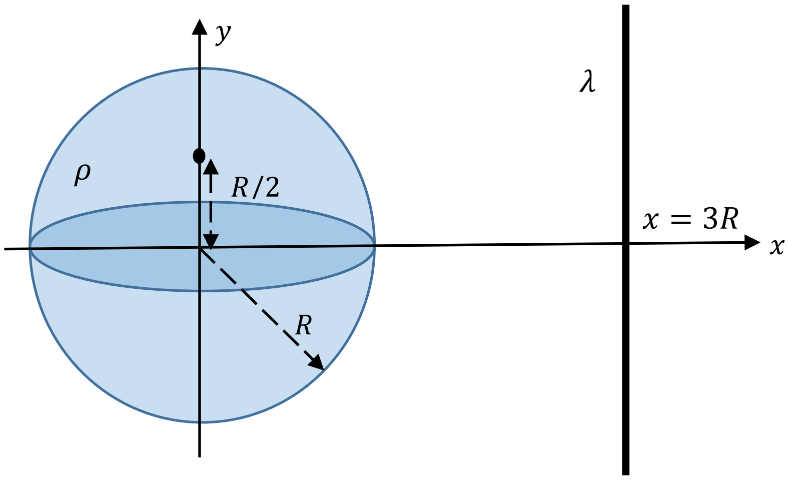
PHYS 102 Midterm Exam 1 Solution 2020-21-2

1. A sphere of radius R is filled uniformly with charge density ρ, and its center is at the origin of the coordinates. An infinite line charge with uniform density λ is parallel to the y-axis and passes through the point $x = 3R, y = 0, z = 0$.

(a) (15 Pts.) Find the magnitude of the electric field at $x = 0, y = R/2, z = 0$.

(b) (10 Pts.) Find the total force the line charge acts on the sphere. Give both direction and magnitude.



Solution:

(a) We use the superposition principle. Magnitude of the electric field created by the uniformly charged sphere at the point $(0, R/2, 0)$ can be found by applying Gauss's law to a spherical surface centered at the origin with radius $r = R/2$.

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad E_\rho(r)(4\pi r^2) = \frac{1}{\epsilon_0}\left(\frac{4}{3}\pi r^3\right)\rho \quad \rightarrow \quad \vec{\boldsymbol{E}}_\rho\left(\frac{R}{2}\right) = \frac{\rho R}{6\epsilon_0}\,\hat{\mathbf{j}}.$$

Magnitude of the electric field created by the line charge at the point $(0, R/2, 0)$ can be found by applying Gauss's law to cylindrical surface with axis on the line charge, radius $r = 3R$, and height L.

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad E_\lambda(r)(2\pi r L) = \frac{1}{\epsilon_0}(\lambda L) \quad \rightarrow \quad \vec{\boldsymbol{E}}_\lambda(3R) = \frac{-\lambda}{6\pi\epsilon_0 R}\,\hat{\mathbf{i}}.$$

Total electric field at that point is the vector sum of the two.

$$\vec{E} = \vec{\boldsymbol{E}}_\rho + \vec{\boldsymbol{E}}_\lambda = \frac{-\lambda}{6\pi\epsilon_0 R}\,\hat{\mathbf{i}} + \frac{\rho R}{6\epsilon_0}\,\hat{\mathbf{j}} \quad \rightarrow \quad E = \frac{R}{6\epsilon_0}\sqrt{\rho^2 + \left(\frac{\lambda}{\pi R^2}\right)^2}.$$

(b) Since both charges are of the same sign, the force is a repulsive one. The electric field created by the sphere at points outside the sphere is the same as that of a point charge centred at the origin. By Newton's third law, magnitude of the electric force exerted by the sphere on the line charge is the same as the magnitude of the electric force exerted by the line charge on the sphere. Therefore, we can replace the sphere by a point charge Q at the origin and write

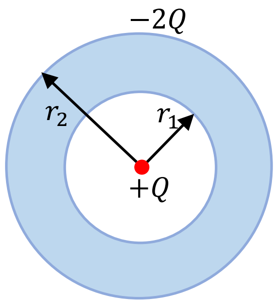
$$F = QE_\lambda = \frac{Q\lambda}{6\pi\epsilon_0 R}.$$

Since the total charge on the sphere is $Q = \left(\frac{4}{3}\pi R^3\right)\rho$, the result is

$$\vec{\boldsymbol{F}} = \frac{2\rho\lambda R^2}{9\epsilon_0}\,(-\hat{\mathbf{i}}).$$

2. (a) (5 Pts.) A conducting spherical shell with inner radius r_1 and outer radius r_2 has a total charge $-2Q$ placed on it. What is the electric potential energy of this charge distribution?

An extra point charge $+Q$ is now placed at the center of the spherical shell described above as shown in the figure.



(b) (5 Pts.) What are the surface charge densities σ_1, and σ_2 on the inner and the outer surfaces of the spherical shell?

(c) 15 Pts.) Find the electric potential $V(r)$ in the regions $0 \leq r \leq r_1$, $r_1 \leq r \leq r_2$ and $r_2 \leq r < \infty$. (Take the electric potential at infinity to be zero.)

Solution:

(a) Suppose we place a charge q on the conducting spherical shell. This charge will be uniformly distributed on the outer surface of the conductor producing a radial electric field for $r > r_2$ same as that of a point charge at the origin. Therefore, the outer surface of the shell will be an equipotential surface with $V(r_2) = q/(4\pi\epsilon_0 r_2)$. The amount of work done to place an extra amount of charge dq on the surface is dW, and

$$dW = V(r_2)dq = \frac{q}{4\pi\epsilon_0 r_2}dq.$$

Hence, the work required to assemble a total of $-2Q$ charge on the surface of the shell, which also is equal to the electric potential energy of this charge distribution, is found as

$$W = \int_0^{-2Q} \frac{q}{4\pi\epsilon_0 r_2}dq = \frac{1}{4\pi\epsilon_0 r_2}\int_0^{-2Q} q dq = \frac{1}{8\pi\epsilon_0 r_2}(-2Q)^2 = \frac{Q^2}{2\pi\epsilon_0 r_2}.$$

(b)

$$\sigma_1 = \frac{-Q}{4\pi r_1^2}, \qquad \sigma_2 = \frac{-Q}{4\pi r_2^2}.$$

(c) The electric field is

$$\vec{\boldsymbol{E}}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}, 0 < r < r_1, \qquad \vec{\boldsymbol{E}} = 0, r_1 < r < r_2, \qquad \vec{\boldsymbol{E}} = \frac{-Q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}, r_2 < r < \infty.$$

Therefore, the electric potential is found as

$$V(\infty) - V(r) = \int_\infty^r \frac{-Q}{4\pi\epsilon_0 r'^2}dr' = \frac{Q}{4\pi\epsilon_0 r} \quad \rightarrow \quad V(r) = \frac{-Q}{4\pi\epsilon_0 r}, r_2 \leq r < \infty.$$

Since $\vec{\boldsymbol{E}} = 0$ in the conductor, the electric potential is constant, and is equal to the potential on the surface. So,

$$V(r) = \frac{-Q}{4\pi\epsilon_0 r_2}, r_1 \leq r \leq r_2.$$

Noting that the potential $V(r)$ is a continuous function, we have

$$V(r_1) - V(r) = \int_{r_1}^r \frac{Q}{4\pi\epsilon_0 r'^2}dr' = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r}\right).$$

Since $V(r_1) = \frac{-Q}{4\pi\epsilon_0 r_2}$, we obtain

$$V(r) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r_1} - \frac{1}{r_2}\right), 0 < r < r_1.$$

3. The side surface of a thin cylindrical shell of length L and radius R is uniformly charged with total charge Q.

a) (5 Pts.) Find the surface charge density σ on the shell.

b) (20 Pts.) Find the electric potential at point P, which is on the symmetry axis at the top point of the cylinder.

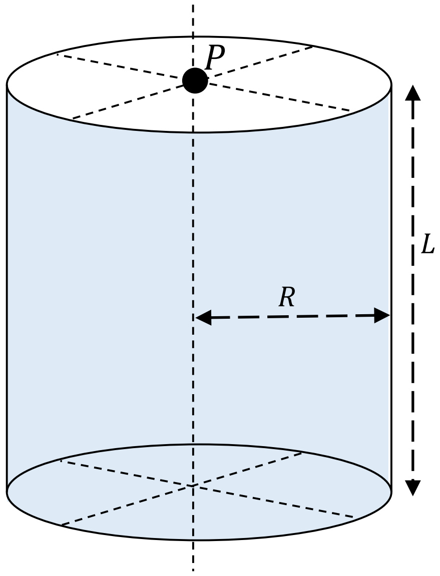
(You may find some of the integrals given below useful.)

$$\int \frac{x}{\sqrt{x^2+d^2}}dx = \sqrt{x^2+d^2}\ , \quad \int \frac{x}{\left(x^2+d^2\right)^{3/2}}dx = \frac{-1}{\sqrt{x^2+d^2}}$$

$$\int \frac{1}{x^2+d^2}dx = \frac{1}{d}\arctan\left(\frac{x}{d}\right) \quad , \int \frac{1}{\sqrt{x^2+d^2}}dx = \ln\left(x+\sqrt{x^2+d^2}\right)$$

$$\int \frac{x^2}{\sqrt{x^2+d^2}}dx = \frac{1}{2}d\sqrt{x^2+d^2} - \frac{1}{2}d^2\ln\left(x+\sqrt{x^2+d^2}\right)$$

$$\int \frac{1}{\left(x^2+d^2\right)^{3/2}}dx = \frac{x}{d^2\sqrt{x^2+d^2}}$$

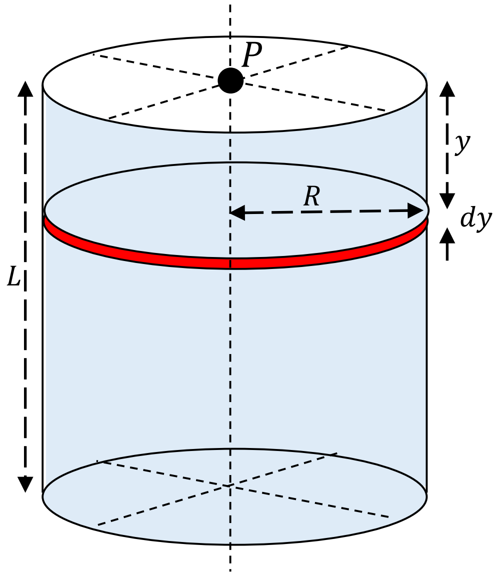


Solution: (a)

$$\sigma = \frac{Q}{A} = \frac{Q}{2\pi RL}.$$

(b) We divide the cylinder into infinitesimal sections each of thickness dy as shown. Surface area of each infinitesimal section is $dA = 2\pi R dy$. Therefore, each section is a ring of charge $dq = \sigma\, dA = 2\pi\sigma R dy$. The electric potential on the symmetry axis of one of these rings at a distance y from its center is

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{\sqrt{y^2+R^2}} = \frac{\sigma R}{2\epsilon_0}\frac{dy}{\sqrt{y^2+R^2}}.$$

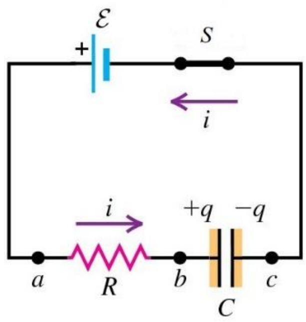


We find the electric potential of the cylinder by integrating over the rings.

$$V = \frac{\sigma R}{2\epsilon_0}\int_0^L \frac{dy}{\sqrt{y^2+R^2}} = \frac{\sigma R}{2\epsilon_0}\ln\left(\frac{L+\sqrt{L^2+R^2}}{R}\right),$$

$$V = \frac{Q}{4\pi\epsilon_0 L}\ln\left(\frac{L}{R}+\sqrt{1+\frac{L^2}{R^2}}\right).$$

4. The figure shows a circuit charging an initially uncharged capacitor. The capacitor being charged is a parallel-plate capacitor with circular plates of radius r and plate separation d. The switch S is closed at time $t = 0$, and the current through the capacitor is given as $i(t) = I_0 e^{-t/\tau}$. Answer the following questions in terms of $r, d, I_0,$ and τ.



(a) (5 Pts.) What is the charge on the capacitor as a function of time?

(b) (5 Pts.) What is the electric field magnitude between the plates as a function of time?

(c) (5 Pts.) Find $\mathcal{E}$ in terms of $r, d, I_0,$ and τ.

(d) (5 Pts.) Find R in terms of $r, d,$ and τ.

(e) (5 Pts.) What is the electrical energy stored in the capacitor in the limit $t \to \infty$?

Solution:

(a)

$$i(t) = \frac{dQ}{dt} \quad \to \quad Q(t) = \int_0^t i(t')dt' \quad \to \quad Q(t) = I_0 \int_0^t e^{-t'/\tau} dt' = I_0 \tau \left(1 - e^{-t/\tau}\right).$$

(b)

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 \pi r^2} \quad \to \quad E(t) = \frac{I_0 \tau}{\epsilon_0 \pi r^2}\left(1 - e^{-t/\tau}\right).$$

(c)

$$\mathcal{E} = \frac{Q_{\text{final}}}{C}, \qquad Q_{\text{final}} = \lim_{t\to\infty} Q(t) = I_0 \tau\,, \qquad C = \epsilon_0 \frac{\pi r^2}{d} \quad \to \quad \mathcal{E} = \frac{I_0 \tau\, d}{\epsilon_0 \pi r^2}.$$

(d)

$$\tau = RC \quad \to \quad R = \frac{\tau}{C} = \frac{\tau d}{\epsilon_0 \pi r^2}.$$

(e)

$$U = \frac{Q^2}{2C} \quad \to \quad U = \frac{(I_0 \tau)^2 d}{2\pi \epsilon_0 r^2}.$$